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Wave Propagation Phenomena from a Spatiotemporal Viewpoint: Resonances and Bifurcations

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By using biorthogonal decompositions, we show how uniformly propagating waves, togehter with their velocity, shape, and amplitude, can be extracted from a spatiotemporal signal consisting of the superposition of various traveling waves. The interaction between the different waves manifests itself in space-time resonances in case of a discrete biorthogonal spectrum and in resonant wavepackets in case of a continuous biorthogonal spectrum. Resonances appear as invariant subspaces under the biorthogonal operator, which leads to closed sets of algebraic equations. The analysis is then extended to superpositions of dispersive waves for which the (Fourier) dispersion relation is no longer linear. We then show how a space-time bifurcation, namely a qualitative change in the spatiotemporal nature of the solution, occurs when the biorthogonal operator is a nonholomorphic function of a parameter. This takes place when two eigenvalues are degenerate in the biorthogonal spectrum and when the spatial and temporal eigenvectors rotate within each eigenspace. Such a scenario applied to the superposition of traveling waves leads to the generation of additional waves propagating at new velocities, which can be computed from the spatial and temporal eigenmodes involved in the process (namely the shape of the propagating waves slightly before the bifurcation). An eigenvalue degeneracy, however, does not necessarily lead to a bifurcation, a situation we refer to as being self-avoiding. We illustrate our theoretical predictions by giving examples of bifurcating and self-avoiding events in propagating phenomena.

KEY WORDS: Wave propagation; spatiotemporal bifurcation theory; biorthogonal decomposition; Fourier analysis.

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1. INTRODUCTION

While temporal dynamical systems theory can describe the motion of individual particles or the dynamics in closed systems whose spatial complexity is severely constrained by the presence of boundaries, there is no doubt that fully spatiotemporal dynamics characterize open and spatially extended systems. While in the former case it suffices to consider the state of the system as a function of time $u(t)$ at any position x_0, in the latter situation this description is no longer sufficient and we need to take into account the full space-time behavior, $u(x, t)$. In this case, there is not much hope, from a theoretical and experimental viewpoint, for a deep understanding of the mechanisms involved, from techniques which freeze space and make time vary or vice versa: a large variety of new phenomena involving a simultaneous evolution in space and time appear, such as wave propagations.

While oscillations are numerous in temporal dynamical systems, waves are ubiquitous in spatiotemporal dynamics. The simplest example is a uniformly traveling wave of any shape (e.g. sinusoidal, solitary), invariant under spatial translation after the appropriate time shift, namely invariant under spatiotemporal translation. Nevertheless, wave motions in nature are not so simple, often consisting of superpositions of waves with various amplitudes and velocities. Even when this superposition is the sum of uniformly traveling waves, the extraction of the individual waves, the determination of their velocity and shape, and the identification of resonances (due to the interactions between the waves) is nontrivial and, to our knowledge, has not been previously resolved. Moreover, it should be understood that the question we raise, together with the answer we propose in this paper, goes much beyond pattern recognition. Our aim is not the extraction of "spatial organized patterns" or "coherent structures," but rather an understanding of the dynamics of $u(x, t)$ in *space and time*, and particularly how the dynamics in space and the dynamics in time are intimately connected. The study of such dynamics leads to the investigation of invariant subspaces under the dynamic operator [whose kernel is simply $u(x, t)$ itself, see below], namely independent portions of the dynamics. Such invariant subspaces are indeed found in this paper, having the physical interpretation of wave resonances. Finally, a complete comprehension of such space-time dynamics should encounter the possibility of bifurcations as a parameter varies. This leads to the next question.

Another issue concerns the route to spatiotemporal complexity. Even when the first instability of a physical phenomenon gives rise to a uniformly traveling wave, subsequent instabilities occurring as a parameter varies often lead to much more complex space-time behavior. The question then arises: how does a traveling wave become unstable and lose its spatiotemporal symmetry? Can one identify (space-time) bifurcations to successive (structurally) stable states before reaching an apparently fully disordered state? For instance, let us consider a wake flow behind a circular cylinder for which the primary instability leads to a Karman vortex street, namely a "street" of vortices propagating downstream (we neglect here the decrease in amplitude due to the viscous dissipation). As Reynolds number increases, the flow clearly deviates from a traveling wave, undergoes deformations as it moves downstream, and eventually becomes turbulent at higher Reynolds numbers. Such a route toward space-time complexity is not yet understood. The same question arises for the surface dynamics of film flow on an incline[28,29] or for shallow water long waves, for which it is known that the (Fourier) dispersion relation is, to a first approximation, linear.[42] When the latter is nonlinear, the difficulty is even increased. This is, for instance, the case of the motion of the water due to a disturbance over a restricted area of the surface, for which it is well known that the propagation speed is an increasing function of the wavelength. Dispersion also occurs in binary fluid convection.[27] Other examples are furnished by partial differential equations (see, e.g., ref. 44). We emphasize here that since we consider the generation of space-time complexity, we concentrate on spatiotemporal bifurcations through which the space-time qualitative nature of the solution is altered: a change in the shape or the propagation speed of a traveling wave, for instance, is not, *a priori*, a space-time bifurcation, in contrast with the appearance of a new traveling wave in the solution. The question we address here is thus different from that treated in ref. 10, where a spatial study (ignoring the time dependence) is carried out to analyze the shape evolution of a uniformly traveling wave. Since a space-time bifurcation, and particularly the appearance of a new propagation speed, is related to the shape of the wave, as we show in this paper, it is possible that the "bifurcations" (in the classical sense of temporal dynamical systems theory) the previous authors identify may be related to space-time bifurcations (in the sense of Section 7 below), but more investigations need to be carried out before such conclusions can be drawn. Moreover, the effect of resonances on the temporal behavior of the waves, such as the generation in time of new traveling waves, as in refs. 20 and 22–24, is a different question than that we treat here.

While in the past, spatiotemporal complexity was treated statistically, our approach is fully deterministic and deals with the "microscopic" space-time function $u(x, t)$ itself, although we recover statistical properties in a natural manner since they correspond to various products of the dynamic operator and its adjoint (two-point correlations are a particular case).[3] Since the space-time symmetries of the spatiotemporal dynamics on which this paper is based manifest themselves on the statistics, one can detect the spatial (resp. temporal) component of the symmetry through the spatial (resp. temporal) two-point correlations, for instance. Whether one can carry out the analysis further and derive a statistical mechanics of wave propagation phenomena is still an open question, although a good "macroscopic" function is the entropy introduced in previous work.[3] We show here how the latter is related to spatiotemporal bifurcations, for instance. We point out, however, that while this quantity may be useful for *detecting* bifurcations, a full space-time analysis is necessary for *understanding* the latter.

In this paper, we address the previous issues by using biorthogonal decompositions which have been proposed and used as tools for studying spatiotemporal dynamics in general.[3,4,7,41] This may appear awkward at first since the Fourier analysis is commonly believed to be the "best" for dealing with propagating phenomena. Let us suppose that $u(x, t)$, defined on a spatial domain X and a temporal interval T, is the spatiotemporal function we would like to study. Biorthogonal decompositions provide the smallest linear subspace $\chi(X)$ containing the orbit $\xi_t(x)$ (described as time runs) defined by

$$\forall x \in X \quad \xi_t(x) = u(x, t)$$

and the smallest linear subspace $\chi(T)$ containing the orbit $\eta_x(t)$ (described as the spatial position varies) defined by

$$\forall t \in T, \quad \eta_x(t) = u(x, t)$$

It follows that these two subspaces adapt themselves to the evolution of the spatiotemporal dynamics. While it is true that biorthogonal temporal and spatial modes coincide with Fourier modes in the case of a uniformly traveling wave, this is not the case, as we show in this paper, as soon as (at least) another traveling wave, even of small amplitude, appears in the signal. The difference between biorthogonal and Fourier modes, even small in certain cases, contains *precisely* the interesting information permitting the detection and identification of resonances between various propagating waves, and provides a tool for analyzing space-time bifurcations of propagating phenomena. It is thus a conceptual mistake, from our viewpoint and for our purpose, to identify *a priori* the two decompositions. Moreover, the fact that spatial biorthogonal modes of a uniformly traveling wave are Fourier modes (which is also necessarily the case for the modes decomposing the spatial two-point correlation and therefore those obtained in the Karhunen–Loève expansion when time is considered as the sampling variable) is often considered as a negative point in previous works where a substitute decomposition is sought in the direction of propagation (see, e.g., refs. 12, 32, and 39). In our work, it is an enormous advantage, along with the fact that temporal modes are also of the Fourier type, since it provides a mean for analyzing the information (chronos and topos) given by the biorthogonal decomposition. It is obvious that if other symmetries than translations are involved (in a superposition process, for instance), other "fundamental" modes should be used for this purpose. However, the direct use of these modes for decomposing the signal is inappropriate for the investigation of the spatiotemporal structure of the signal and its space-time bifurcations. Particularly for the classification and prediction of the latter (see Section 7 below), the isomorphism between the subspaces $\chi(X)$ and $\chi(T)$ (characterizing biorthogonal decompositions) is necessary.

We now recall that biorthogonal decompositions consist in expanding a function $u(x, t)$ (where usually x and t denote space and time variables) into orthogonal modes in a Hilbert space $H(X)$ ($x \in X$) and orthogonal modes in a Hilbert space $H(T)$ ($t \in T$) which are related by a one-to-one correspondence. Such expansions correspond to the spectral decompositions of the operators V:

$$V = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}$$

where U, which acts from $H(X)$ to $H(T)$, is defined as

$$\forall \varphi \in H(X), \qquad (U\varphi)(t) = \int_X u(x, t)\, \varphi(x)\, dm(x)$$

the adjoint operators U^* from $H(T)$ to $H(X)$ being such that

$$\forall \psi \in H(T), \qquad (U^*\psi)(x) = \int_T \overline{u(x, t)}\, \psi(t)\, d\tilde{m}(t)$$

where $dm(x)$ [resp. $d\tilde{m}(t)$] denotes the measure defining the scalar product in $H(X)$ [resp. $H(T)$] and the bar refers to the complex conjugate. If V is a compact operator, the corresponding biorthogonal decomposition can be written as

$$u(x, t) = \sum_{n=1}^{N} A_n \overline{\varphi_n(x)}\, \psi_n(t)$$

with $A_1 \geqslant A_2 \geqslant \cdots A_N > 0$ and $(\varphi_n, \varphi_m) = (\psi_n, \psi_m) = \delta_{n,m}$, which converges in norm. The parentheses here denote the scalar products in $H(X)$ and $H(T)$. Hereafter, the spatial eigenmodes φ_n are called *topos* and the temporal eigenmodes ψ_n are called *chronos*. The isomorphism between a topo and a chrono is given by the operator U itself, that is, $U\varphi_n = A_n \psi_n$. The spatiotemporal dynamics can then be studied in the temporal configuration space $\chi(T) = \text{Ker}(U^*)^\perp$ spanned by the chronos and the spatial one $\chi(X) = \text{Ker}(U)^\perp$ spanned by the topos. Such a decomposition is obviously not new, it originates in the spectral decomposition of operators with symmetric kernels, which can be found early in textbooks[16] for compact operators and for operators with Carleman kernels in ref. 43. As we mentioned above, we recall that the probability theory tool referred to as the Karhunen–Loève expansion,[24,31] in which the sampling variable is time, corresponds to the spectral decomposition of U^*U, whose kernel is the spatial two-point correlation. It has been proposed by Lumley for its application to turbulence[32,33] (see also ref. 11) and applied in a number of studies (see, e.g., refs. 6, 9, 17, 19, 26, 38, 39, and 40 and references therein). In our work, it is fundamental to consider the operator V for a simultaneous treatment of space and time. Moreover, the introduction of general Hilbert spaces $H(X)$ and $H(T)$ along with the consideration of noncompact operators are fundamental when spectra are continuous, as we show in Section 6 (see also ref. 5).

In this context, space-time symmetries have been introduced[4] as a pair of operators $(\tilde{S}, S)$ (or a group of such operators), $\tilde{S}$ acting on $H(T)$ and S on $H(X)$, which intertwine the operator U, namely $\tilde{S}U = US$, and whose conjugate pair $(\tilde{S}^*, S^*)$ also satisfies $\tilde{S}^*U = US^*$. This was shown to be equivalent to a degeneracy of the topos and chronos, in the sense that if $U\varphi_n = A_n \psi_n$, then $US\varphi_n = A_n \tilde{S}\psi_n$. It is interesting to note that such spatiotemporal symmetry manifests itself in the statistics recovered by considering various powers of the operator V. The second-order statistics are the kernels of the operators UU^* and UU^* (obtained from V^2), which, in case of a spatiotemporal symmetry, commute with $\tilde{S}$ and S, respectively: $\tilde{S}UU^* = UU\tilde{S}$ and $SU^*U = U^*US$. An example of such symmetry is satisfied by a traveling wave of velocity c.[4] In this case, the spatial and temporal symmetry operators involved are regular representations of R, namely translations:

$$(S_{x_0}\varphi)(x) = \varphi(x - x_0)$$

$$(\tilde{S}_{t_0}\psi)(t) = \psi(t - t_0)$$

such that $x_0 + ct_0 = 0$, making all topos and chronos Fourier modes, and the spectrum degenerate, of order two. Other examples involving the group of dilations and noncompact operators have been used for the treatment of fully developed (incompressible and compressible) turbulence[5,8] (see also ref. 2 for the general use of biorthogonal decompositions in turbulence).

2. FOURIER TRANSFORM AND BIORTHOGONAL DECOMPOSITIONS

The use of the Fourier transform in the spectral analysis of linear operators in a Hilbert space is a well-known subject (see, e.g., ref. 36). Here, we limit ourselves to the minimum necessary for defining some notations and terminology and for clarifying how we utilize these notions in the context of biorthogonal decompositions.

On the one hand, the Fourier analysis of $u(x, t)$, $x \in X$, $t \in T$ when X and T are bounded domains and $u(x, t)$ is spatially and temporally periodic, can be viewed as a decomposition which uses specific orthonormal bases of $L^2(X)$ and $L^2(T)$.

On the other hand, one may consider the Fourier transforms as two linear operators F and $\tilde{F}$ as in the present study. The spatial Fourier transform F,

$$(F\varphi)(k) = c' \int_X \varphi(x)\, e^{ikx}\, dx \tag{2.1}$$

is defined as an operator from $L^2(X)$ to $L^2(K)$, where K is the domain of wavenumbers ($K = \hat{X}$), namely integers if X is a bounded domain and R if $X = R$. Similarly, the temporal Fourier transform is defined from $L^2(T)$ to $L^2(\Omega)$, where Ω is the frequency domain ($\Omega = \hat{T}$), such that

$$(\tilde{F}\psi)(\omega) = c'' \int_T \psi(t)\, e^{-i\omega t}\, dt \tag{2.2}$$

The coefficients introduced in the definitions (2.1) and (2.2) of the spatial and temporal Fourier transforms are normalization constants so that the operators F and $\tilde{F}$, by the Parseval theorem, are unitary. Therefore, the spectral analysis of the operator U and that of the operator

$$\hat{U} = \tilde{F} U F^{-1} \tag{2.3}$$

are equivalent. More precisely, if

$$u(x, t) = \int A \varphi_A(x)\, \psi_A(t)\, dm(A) \tag{2.4}$$

for m-almost all A, is the biorthogonal decomposition of $u(x, t)$ corresponding to the spectral analysis of the operator U, dm being the spectral measure, then the corresponding biorthogonal decomposition of the space-time Fourier transform $\hat{u}(k, \omega)$ is

$$\hat{u}(k, \omega) = \int A \hat{\varphi}_A(k)\, \hat{\psi}_A(\omega)\, dm(A) \tag{2.5}$$

for m-almost all A, where

$$\hat{\varphi}_A = F\varphi_A \tag{2.6}$$

$$\hat{\psi}_A = \tilde{F}\psi_A \tag{2.7}$$

In particular, if U has only a point spectrum (eigenvalues), so does $\hat{U}$ and the biorthogonal decompositions of u and $\hat{u}$ are related by the Fourier transforms of the topos and chronos. In other words, if

$$u(x, t) = \sum_n A_n \varphi_n(x)\, \psi_n(t) \tag{2.8}$$

then

$$\hat{u}(k, \omega) = \sum_n A_n \hat{\varphi}_n(k)\, \hat{\psi}_n(\omega) \tag{2.9}$$

Now, if $u(x, t)$ is a traveling wave, namely

$$u(x - x_0, t) = u(x, t + t_0) \tag{2.10}$$

for all $x, x_0 \in X$, t, $t_0 \in T$, such that $x_0 + ct_0 = 0$, c being the propagation speed of the wave, also called the velocity of the wave, then $\hat{U}$ is a multiplication operator, i.e., $\tilde{F} \cdot F^{-1}$ "diagonalizes" U, and the spectral analysis of U is given by the Fourier analysis, namely the biorthogonal decomposition (2.4) of $u(x, t)$ coincides with the Fourier transform.[4]

In particular, if U has only a point spectrum as is the case if X and T are bounded domains, then $\hat{\varphi}_n$ and $\hat{\psi}_n$ in (2.9) are delta functions. Then, considering a traveling wave with a shape g such that

$$u(x, t) = g(x - ct) \tag{2.11}$$

for all $x \in X$ and all $t \in T$, where c is the wave velocity, we derive the well-known localization property of the Fourier transform $\hat{u}$:

$$\begin{aligned} \hat{u}(k, \omega) &= \hat{g}(k) \qquad \text{if} \quad \omega = ck \\ &= 0 \qquad\quad \text{otherwise} \end{aligned} \tag{2.12}$$

Obviously, in general, $\hat{U} = \tilde{F}UF^{-1}$ is not a "diagonal" operator and therefore the biorthogonal decomposition of $u(x, t)$ does not coincide with the Fourier decomposition. This is the case when, for instance, $u(x, t)$ is the superposition of two (or more) traveling waves of different speeds, which we treat in the next section. Then, as Fourier wrote, "we have therefore ... a problem whose solution demands attentive examination."

3. SUPERPOSITION OF TWO TRAVELING WAVES AND RESONANCES

We illustrate our technique by first treating the case of the superposition of two traveling waves, u_1 with propagation speed c_1 and u_2 with propagation speed c_2, such that $|c_1| < |c_2|$.

Since the case of continuous spectra will be treated in Section 6, we restrict ourselves to the situation where u_1 and u_2 are defined in a common bounded space-time domain, extended by periodicity. In this case, c_1 and c_2, when normalized by the ratio between the length of the spatial domain and that of the temporal interval, are necessarily rational numbers. Let us then set $c_1 = p_1/q_1$ (resp. $c_2 = p_2/q_2$), where p_1 and q_1 (resp. p_2 and q_2) are incommensurate. The eigenequations for the operator $\hat{U}_1 + \hat{U}_2$ then become

$$(\hat{U}_1 + \hat{U}_2)\hat{\varphi} = A\hat{\psi} \tag{3.1a}$$

$$(\hat{U}_1 + \hat{U}_2)^*\hat{\psi} = A\hat{\varphi} \tag{3.1b}$$

The first (resp. the second) of these vectorial equations can be written as an infinite set of scalar equations, one for each frequency $\omega \in \Omega$ (resp. each wavenumber $k \in K$). Let us denote g_1 and g_2 the shapes of the traveling waves $u_1(x, t)$ and $u_2(x, t)$ [defined by (2.11)] for which we have the localization property (2.12) in Fourier space. It is clear that the equations corresponding to (3.1a) can take one of the following forms only: either

$$\hat{g}_i(k)\,\hat{\varphi}_k = A\hat{\psi}_\omega \tag{3.2}$$

if $\omega = ap_i$, $a \in \mathbb{Z}$, $i = 1$ or $i = 2$, and then $k = aq_i$; or

$$\hat{g}_1(k_1)\,\hat{\varphi}_{k_1} + \hat{g}_2(k_2)\,\hat{\varphi}_{k_2} = A\hat{\psi}_\omega \tag{3.3}$$

if $\omega = ap_1 = bp_2$, a, $b \in \mathbb{Z}$, and then $k_1 = aq_1$ and $k_2 = bq_2$. Similarly, the equations corresponding to (3.1b) takes only one of the following forms: either

$$\hat{g}_i(k)\,\hat{\psi}_\omega = A\hat{\varphi}_k \tag{3.4}$$

if $k = aq_i$, $a \in \mathbb{Z}$, $i = 1$ or $i = 2$, and then $\omega = ap_i$, or

$$\hat{g}_1(k)\,\hat{\psi}_{\omega_1} + \hat{g}_2(k)\,\hat{\psi}_{\omega_2} = A\hat{\varphi}_k \tag{3.5}$$

if $k = aq_1 = bq_2$, a, $b \in \mathbb{Z}$, and then $\omega_1 = ap_1$ and $\omega_2 = bp_2$.

Due to the presence of equations of the type (3.3) and (3.5), a closed subset of equations (3.2)–(3.5) may require more than two equations. We also notice that some frequencies ω or wavenumbers k appearing in (3.2) [resp. (3.4)] may also appear in (3.5) [resp. (3.3)], thus forming a *chain* corresponding to a closed subset of equations. Of course, some particular properties of the shapes g_1 and g_2 may introduce additional decouplings of the eigenequations since some of the Fourier transforms $\hat{g}_1(k)$ and $\hat{g}_2(k)$ may be zero. However, we consider the *generic* situation where this is not the case (see both our comment at the end of this section and the second example of Section 8 for a nongeneric situation). The appearance of chains of frequencies and wavenumbers leads us to introduce the notion of a spatiotemporal resonance.

Definition 3.1. A spatiotemporal resonance is a maximal set of integers $\omega_1, \omega_2, \ldots, \omega_n$, $k_1, k_2, \ldots, k_m$, n and m being finite or infinite, such that

$$\omega_i = c_\alpha k_i \tag{3.6}$$

and

$$\omega_{i\pm 1} = c_\beta k_i \tag{3.7}$$

where either $\alpha = 1$ and then $\beta = 2$, in which case $i+1$ should be selected in (3.7), or $\alpha = 2$ and then $\beta = 1$, in which case $i-1$ should be selected in (3.7). Here, "maximal" has the usual meaning of characterizing a set which cannot be properly included in a larger one satisfying (3.6) and (3.7). Due to the chain rule used to go from (3.6) to (3.7) and back, a spatiotemporal resonance satisfies the two following properties:

(i) The frequencies ω_i and the wavenumbers k_i are in a strictly increasing order of their absolute values (since $|c_1| < |c_2|$).

(ii) The number n of frequencies and the number m of wavenumbers are simultaneously either finite or infinite.

When the number n of frequencies and the number m of wavenumbers are finite, n and m are related by the relation

$$m = n \pm 1$$

We define the (possibly infinite) order of a spatiotemporal resonance by the smallest number among n and m. Before justifying the terminology "resonance" we explain how spatiotemporal resonances appear in the spectral analysis of $U_1 + U_2$. Hereafter, we refer to the sets of frequencies and wavenumbers introduced in this definition as "resonances."

Lemma 3.2. There is a one-to-one correspondence between the closed set of eigenequations (3.2)–(3.5) and resonances.

Proof. By easy inspection. Notice that for the sake of completeness, we need to include the "resonances" consisting of only one frequency ω and one wavenumber k which realize the closure with only one equation of the type (3.2) and one equation of the type (3.4). In this case, the set ω, k is maximal without linking U_1 to U_2, so that the terminology "resonance" is not quite appropriate. We refer to this particular set as a "degenerate resonance."

Due to their maximal property (in the sense of Definition 3.1), resonances perform a partition of Ω and K so that each ω (resp. each k) belongs to one and only one resonance. Another remark is that for real functions g_1 and g_2 the usual symmetry of the Fourier transform implies that only half of the equations in (3.2)–(3.5) are needed to solve the eigenproblem [those corresponding to (ω, k), $(\omega, -k)$, where ω and k are both positive, for instance].

We now show how a resonance is related to the spectral analysis of $U_1 + U_2$ by introducing a pair of orthogonal projectors $(P_{\tilde{\delta}}, P_\delta)$ for each resonance. $P_{\tilde{\delta}}$, acting on $L^2(\Omega)$, projects onto the subspace generated by the delta functions involving the frequencies ω of a given resonance, and P_δ, acting on $L^2(K)$, projects onto the subspace generated by the delta functions involving the corresponding wavenumbers k (of the same resonance). We can then write a new lemma which essentially translates the content of Lemma 3.2 in the language of the spectral analysis of linear operators, a useful step toward the generalizations treated in the following sections.

Lemma 3.3. The set of projections $\{P_{\tilde{\delta}}\}$ in $L^2(\Omega)$ and $\{P_\delta\}$ in $L^2(K)$, where $\tilde{\delta} = \{\omega_1, \omega_2, ..., \omega_n\}$ and $\delta = \{k_1, k_2, ..., k_m\}$ run in the set $\mathfrak{R}$ of all resonances, are two resolutions of the identity such that

$$P_{\tilde{\delta}}(\hat{U}_1 + \hat{U}_2) = (\hat{U}_1 + \hat{U}_2)\, P_\delta \tag{3.8}$$

Proof. It suffices to combine Lemma 3.2 with the eigenequations (3.1a) and (3.1b). We thus obtain

$$\hat{U}_1 + \hat{U}_2 = \sum_{\mathfrak{R}} P_{\tilde{\delta}}(\hat{U}_1 + \hat{U}_2)\, P_\delta \tag{3.9}$$

which reduces the operator $\hat{U}_1 + \hat{U}_2$ in blocks, but, of course, the eigenvectors are not delta functions, since Eqs. (3.2)–(3.5) cannot be decoupled inside a resonance.

Before describing the solution of the eigenequations, we classify all possible resonances in a theorem.

Theorem 3.4. According to the ratio c_1/c_2, we have one of the following situations:

(i) If $c_1 = -c_2$, then there is an infinite number of resonances, each resonance being of finite order (in fact, of order 2).

(ii) If $c_1 = nc_2$, which $n \in \mathbb{Z}$ and $|n| > 1$, then there is an infinite number of resonances, each resonance being of infinite order.

(iii) If c_1/c_2 is not an integer, there is an infinite number of resonances, each resonance being of finite order. In this case, resonances of all finite orders appear.

Proof. Case (i) is the usual case, since a resonance merely consists of $\{\omega, -\omega, k, -k\}$ with $\omega = ck$. Moreover, in this case, there is no degenerate resonances.

Cases (ii) and (iii) are treated in the same manner, namely, from Eqs. (3.6) and (3.7), we write

$$\omega_i = a_i p_1 = b_i p_2 \tag{3.10}$$

$$k_i = a_i q_1 = b_{i+1} q_2 \tag{3.11}$$

where the coefficients a_i and b_i are integers. In addition, we suppose $\omega_1 = c_1 k_1$ (the treatment of the other case, i.e., $\omega_1 = c_2 k_1$, is similar). Note that, because of the incommensurate representation of c_1 and c_2, (3.10) and (3.11) are equivalent to (3.6) and (3.7). We then obtain the values of the integers a_i and b_i

$$a_i = a_1 \left(\frac{c_2}{c_1}\right)^{i-1} \tag{3.12}$$

and

$$b_{i+1} = a_1 \left(\frac{c_2}{c_1}\right)^{i-1} \frac{q_1}{q_2} \tag{3.13}$$

and subsequently the values of ω_i and k_i [since (3.12) and (3.13) are equivalent to the existence of a resonance given by (3.10) and (3.11)]:

$$\omega_i = a_1 \left(\frac{c_2}{c_1}\right)^{i-1} p_1 \tag{3.14}$$

and

$$k_i = a_1 \left(\frac{c_2}{c_1}\right)^{i-1} q_1 \tag{3.15}$$

provided that (3.12) and (3.13) define integers a_i, b_{i+1}.

In case (ii), since $c_2/c_1 = n$, either a_1 is a multiple of $p_1 p_2 q_1 q_2$ and then we get a resonance of infinite order because (3.12) and (3.13) produce integers a_i and b_{i+1} for all i, or a_1 is not a multiple of $p_1 p_2 q_1 q_2$, in which case we have a degenerate resonance (ω_1, k_1). For an infinite resonance, the frequencies ω_i and wavenumbers k_i both increase in a geometric sequence.

In case (iii) we have $c_2/c_1 = p/q$ where again we have chosen an incommensurate pair (p, q). It is then easy to see that a necessary and sufficient condition for obtaining an integer a_i (resp. b_{i+1}) from Eq. (3.12) [resp. (3.13)] is that a_1 is a multiple of q^{i+1} (resp. αq^{i-1}, where α is independent of the exponent i). Therefore, since q^i tends to infinity when i tends to infinity, for any a_1 (and therefore for any resonance), there exists an integer i_0 such that if $i_0 > i$ then the condition cannot be fulfilled. This gives an upper bound for the order of the resonance. Conversely, we may construct resonances of any (finite) order by taking $a_1 = q_2 q^n$. Apart from a slight modification of the notation in Eqs. (3.2)–(3.5), the case where one of the traveling waves has zero or infinite velocity (corresponding to a steady or spatially uniform wave) has nothing special (see Remark 3.6 at the end of this section for a comment on this situation).

Due to the block decomposition of $\hat{U}_1 + \hat{U}_2$ by resonances, we may now perform the complete spectral analysis of the sum of two traveling waves and even reconstruct each traveling wave from the biorthogonal decomposition of $u_1 + u_2$. This is performed via the following theorem.

Theorem 3.5. A spectral subspace of $\hat{U}_1 + \hat{U}_2$ of dimension $2n$ is associated with each resonance of order n (n can be finite or infinite). The corresponding $2n$ topos and $2n$ chronos have a Fourier spectrum whose support is included in the set of wavenumbers and frequencies of the resonance. Moreover, a given wavenumber k_i (resp. a given frequency ω_i) cannot appear in two different resonances. The eigenvalues corresponding to a resonance of finite order are the roots of the corresponding characteristic polynomial of degree $2n$. In this subspace, the wave has a temporal period τ and a spatial period γ (normalized with the size of the temporal and spatial domains) defined by

$$\tau = 1 \Big/ \sum_i \omega_i \tag{3.16}$$

$$\gamma = 1 \Big/ \sum_i k_i \tag{3.17}$$

Furthermore, given the elements of the biorthogonal decomposition, Eqs. (3.2)–(3.5) define an iterative algorithm which permits the reconstruction of the two traveling waves.

Proof. The relation between resonances and spectral subspaces is given by Lemma 3.3. Notice that, even if $\delta = \{k_1,..., k_i,...\}$ and $\tilde{\delta} = \{\omega_1,..., \omega_i,...\}$ do not have the same cardinality, because $U_1 + U_2$ is an isomorphism between the spectral subspaces, the number of topos in $\hat{P}_\delta L^2(X)$ and the number of chronos in $\hat{P}_{\tilde{\delta}} L^2(T)$ are necessarily the same. This dimension of the subspaces is equal to $\min(2n, 2m)$, i.e., twice the order of the resonance. All the vectors in $\hat{P}_{\tilde{\delta}} L^2(T)$, in particular the chronos, have a Fourier spectrum whose support is included in $\tilde{\delta} = \{\omega_1,..., \omega_i,...\}$ and all the vectors in $\hat{P}_\delta L^2(X)$, in particular the topos, have a Fourier spectrum whose support is included in $\delta = \{k_1,..., k_i,...\}$. The fact that we have associated a space of dimension $2n$ to a resonance of order n is simply due to the symmetry $(\omega, k) \to (-\omega, -k)$ when u_1 and u_2 are real-valued. By Lemma 3.3, we can then find the eigenvalues of $U_1 + U_2$ by restricting $\hat{U}_1 + \hat{U}_2$ to each subspace corresponding to a resonance. If the order n of the resonance is finite, the associated subspace is defined by a $2n \times 2n$ symmetric matrix and the result concerning the eigenvalues follows.

Now, let $\varphi_\delta^{(1)},..., \varphi_\delta^{(1)},...$ and $\psi_{\tilde{\delta}}^{(1)},..., \psi_{\tilde{\delta}}^{(1)},...$ be the topos and chronos of a resonance. We can then write

$$\varphi_\delta^{(l)}(x)\, \varphi_{\tilde{\delta}}^{(l)}(t) = \sum_j \hat{\varphi}_\delta^{(l)}(j) \exp(ik_j x) \sum_{j'} \hat{\psi}_{\tilde{\delta}}^{(l)}(j') \exp(-i\omega_{j'} t) \tag{3.18}$$

where the $\hat{\varphi}_\delta^{(l)}$ and the $\hat{\psi}_{\tilde{\delta}}^{(l)}$, $l = 1,...,$ are the solutions of the system of equations (3.2)–(3.5) corresponding to a given resonance $\delta = \{k_1,..., k_j,...\}$, $\tilde{\delta} = \{\omega_1,..., \omega_{j'},...\}$. Therefore, the restriction of the sum of the traveling waves $u_1 + u_2$ to this resonance becomes

$$u_{(\delta,\delta')}(x, t) = \sum_l A_{\delta,\delta'}^{(l)} \sum_j \hat{\varphi}_\delta^{(l)}(j) \exp(ik_j x) \sum_{j'} \hat{\psi}_{\tilde{\delta}}^{(l)}(j') \exp(-i\omega_{j'} t) \tag{3.19}$$

from which the periods (3.16) and (3.17) are immediately deduced. The corresponding orbit in the phase space is a closed Lissajous curve, as expected for a resonance.

We now show how one can easily reconstruct the traveling waves u_1 and u_2 from the biorthogonal decomposition of the superposed signal. For this we note that within each resonance there is always an equation of the type (3.2) or (3.4) with only one frequency ω and one wavenumber k. If $\omega_1 = c_1 k_1$, this equation is

$$\hat{g}_1(k_1)\, \hat{\psi}_{\omega_1} = A \hat{\varphi}_{k_1} \tag{3.20}$$

and if $\omega_1 = c_2 k_1$ it is

$$\hat{g}_2(k_1)\, \hat{\varphi}_{k_1} = A \hat{\psi}_{\omega_1} \tag{3.21}$$

We immediately deduce $\hat{g}_1(k_1)$ in the first case and $\hat{g}_2(k_1)$ in the second case. We now suppose that we are in the first case, namely $\omega_1 = c_1 k_1$. We then proceed with a recurrence formula in which an equation of type (3.3) alternates with an equation of type (3.5):

$$\hat{g}_2(k_i) = A\hat{\psi}_{\omega_{i-1}}/\hat{\varphi}_{k_i} - \hat{g}_1(k_{i-1})\,\hat{\varphi}_{k_{i-1}}/\hat{\varphi}_{k_i} \tag{3.22}$$

and

$$\hat{g}_1(k_i) = A\hat{\varphi}_{k_i}/\hat{\psi}_{\omega_{i-1}} - \hat{g}_2(k_i)\,\hat{\psi}_{\omega_i}/\hat{\psi}_{\omega_{i-1}} \tag{3.23}$$

for $i = 1, 2, \ldots$. The fact that these equations are ordered by increasing wavenumber k_i is important. It implies that the recurrence can run without taking into account its partition among the various resonances, even if the latter can be easily identified from the biorthogonal decomposition by inspecting the Fourier spectrum of the topos and chronos, as indicated in the theorem. Of course, there is an exception in the previous algorithm. At zero wavenumber and zero frequency, Eqs. (3.2) and (3.4) are degenerate, giving rise to one eigenvalue only

$$A = \hat{g}_1(0) + \hat{g}_2(0)$$

The case $\omega_1 = c_2 k_1$ can be treated exactly in the same manner. This ends the proof of Theorem 3.5.

So far we have considered the generic situation where the Fourier transforms $\hat{g}_1(k)$ and $\hat{g}_2(k)$ are nonzero for all k. We now investigate what is changed in the previous analysis if some of the amplitudes are zero, which can happen, for instance, as a consequence of some symmetry. Suppose that a certain $\hat{g}_i(k) = 0$, $i = 1$ or $i = 2$. If $(k, \omega = c_i k)$ is a degenerate resonance, then nothing special happens, since the two corresponding eigenequations (3.2) and (3.4) disappear and the associated Fourier transforms of the topos and chronos $\hat{\varphi}_k$ and $\hat{\psi}_\omega$ belong to the kernel of $\hat{U}$ and $\hat{U}^*$. If, instead, such a pair $(k,\ \omega = c_i k)$ belongs to a (nondegenerate) resonance, we have to distinguish two cases:

(i) If the given (k, ω) is in the first position or last position in the chain, then the situation is the same as in the case of a degenerate resonance.

(ii) Otherwise, (k, ω) splits the resonance into two independent chains, or subresonances, the reason being that, if $\hat{g}_i(k) = 0$, there is a splitting of the eigenequations into two independent closed sets of the type (3.2)–(3.5).

Remark 3.6. Theorem 3.4 is valid in the case where one of the velocities, for instance c_2, is zero (steady wave) or infinite (spatially uniform wave) and we should mention that nothing is really simplified in this situation. This implies that in the case of two waves traveling at non-zero velocities, the difficulties cannot be avoided by adopting the frame of reference of one of the traveling waves. In particular, when one of the waves has zero velocity ($c_2 = 0$), not only do the eigenequations not split into closed subsets of equations, but also there is a resonance of infinite order, as we now show by writing the system of eigenequations:

$$[\hat{g}_1(0) + \hat{g}_2(0)]\,\hat{\varphi}_0 + \sum_{k \neq 0} \hat{g}_2(k)\,\hat{\varphi}_k = A\hat{\psi}_0 \tag{3.24}$$

$$\hat{g}_1(k)\,\hat{\varphi}_k = A\hat{\psi}_\omega, \qquad k \neq 0 \quad \text{and} \quad \omega = c_1 k \tag{3.25}$$

$$(\hat{g}_1(0) + \hat{g}_2(0))\,\hat{\psi}_0 = A\hat{\varphi}_0 \tag{3.26}$$

$$\hat{g}_1(k)\,\hat{\psi}_\omega + \hat{g}_2(k)\,\hat{\psi}_0 = A\hat{\varphi}_k, \qquad k \neq 0 \quad \text{and} \quad \omega = c_1 k \tag{3.27}$$

A consequence of the presence of the resonance of infinite order is that topos and chronos will show all the wavenumbers and frequencies appearing in the problem unless some additional symmetry is assumed, such as, for instance, some (nongeneric) restriction on the shape of u_1 and u_2.

4. SUPERPOSITION OF *N* TRAVELING WAVES

The case of superposition of more than two traveling waves does not introduce any new difficulty besides the fact that in practice the study of the arithmetic relations between the frequencies ω_i and the wavenumbers k_i describing a resonance becomes rapidly cumbersome.

Let us consider the superposition of N traveling waves

$$u(x, t) = \sum_{d=1}^{N} u_d(x, t) = \sum_{d=1}^{N} g_d(x - c_d t) \tag{4.1}$$

of propagation speeds c_d and shapes g_d, $d = 1, 2,..., N$. For the sake of simplicity, we introduce an order among the frequencies ω and the wavenumbers k inside a resonance, even if this knowledge can be recovered from the determination of the sequences of wavenumbers $\{k_1,..., k_i,...\}$ and frequencies $\{\omega_1,..., \omega_i,...\}$, and the velocities c_d.

Definition 4.1. A resonance is a maximal set of integers

$$\{\omega_1, k_1,..., \omega_i, k_i,...\} \tag{4.2}$$

or

$$\{k_1, \omega_1,..., k_i, \omega_i,...\} \tag{4.3}$$

such that the first two elements are necessarily a frequency ω_1 and a wavenumber k_1. After this first pair, two (or more) frequencies and/or two (or more) wavenumbers can be consecutive, but they necessarily satisfy the two following points:

(i) The frequencies ω_i, on the one hand, and the wavenumbers k_i, on the other hand, are strictly increasing sequences, namely

$$\omega_{i_1} < \omega_{i_2} \quad \text{if} \quad i_1 < i_2 \tag{4.4}$$

and

$$k_{j_1} < k_{j_2} \quad \text{if} \quad j_1 < j_2 \tag{4.5}$$

(ii) For each k_j following ω_i and each ω_i following k_j in the chain of a resonance [we say that k_j (resp. ω_i) follows ω_i (resp. k_j) if there is no other ω (resp. k) between k_j (resp. ω_i) and ω_i (resp. k_j)], the frequency and the wavenumber are related by the relation

$$\omega_i = c_d k_j \tag{4.6}$$

where $d = 1,..., N$ depends upon i and j. As in the case of two traveling waves, we refer to the "order of a resonance" (which may be finite or infinite) as the smallest number among the number of wavenumbers k and the number of frequencies ω of the resonance.

We now add some remarks to this definition.

Remark 4.2. 1. The condition (i) of Definition 4.1 implies that that the speed c_d arising in Eq. (4.6) strictly increases for a sequence of frequencies which follows a given wavenumber k and strictly decreases for a sequence of wavenumbers which follows a given frequency ω.

2. A sequence of frequencies ω between two wavenumbers or a sequence of wavenumbers between two frequencies has at most $N-1$ elements (N being the total number of individual traveling waves). As a consequence, the number of frequencies and the number of wavenumbers in a resonance are both either finite or infinite.

3. From the previous point of this remark we can see that in the case of the superposition of two traveling waves ($N = 2$) the frequencies and wavenumbers in a resonance alternate:

$$\omega_1, k_1, \omega_2, k_2,... \tag{4.7}$$

or

$$k_1, \omega_1, k_2, \omega_2,... \tag{4.8}$$

and therefore in this case (4.6) is equivalent to (3.6) and (3.7).

4. For $N > 2$, a given resonance may not include the N propagation speeds $c_1, \dots, c_N$ present in the superposed function $u(x, t)$, since we can build a resonant sequence with a smaller number of velocities. Obviously, at least two velocities are, however, needed, unless we deal with a degenerate resonance with only one frequency ω and one wavenumber k.

We now follow the same procedure as for the case of the superposition of two traveling waves ($N = 2$). Since we get the same results with similar proofs, we avoid going into details. Instead, we prefer to add a comment which may be useful in practice. When one performs the biorthogonal decomposition of (4.1), resonances are studied by examining the Fourier spectrum of the topos and chronos: a resonance is detected by the existence of Fourier peaks located at the same wavenumber in various topos and at the same frequency in various chronos. Again, the number of such topos and chronos is twice the order of the resonance. Since the velocities are not known at this point, the ordering of the wavenumbers k and the frequencies ω inside a resonance may be a delicate task for large resonances. This is why it is always better to study the low-order resonances first and determine the velocities before proceeding with higher-order resonances.

Moreover, it is worth mentioning that the biorthogonal decomposition makes a clear distinction between the energies of the various spatiotemporal (biorthogonal) modes, i.e., topos and chronos, of the signal (given by the square of the eigenvalues of the operator U) and the spatial and temporal Fourier amplitudes of the topos and chronos which are related to their orientation with respect to the Fourier basis.

Remark 4.3. Suppose that we are only interested in identifying, in a given signal, the number of traveling waves and their velocities. One may consider the technique proposed by ref. 35, consisting in performing a first Fourier transform to obtain, for instance, $\hat{u}(k, t)$, selecting a wavenumber $k = k_0$, and performing the temporal Fourier transform of $\hat{u}(k_0, t)$. The frequencies ω thus detected will indicate the possible existence of traveling waves of velocities ω/k_0. The issue is then the following: is it possible, in this manner, to detect all traveling waves, or, equivalently, does k_0 exist and, in the case of existence, how can we select the appropriate k_0? Note that if a temporal Fourier transform is performed first, requiring the selection of a given frequency, the same question arises. Clearly, the only wavenumbers k_0 (resp. frequency ω_0) which will lead to the resolution of this problem are those for which the Fourier transform $\hat{u}(k_0, t)$ [resp. $\hat{u}(x, \omega_0)$] contains *all* possible frequencies. This implies that k_0 (resp. ω_0) must be present in a resonance of *all* the traveling waves present in the signal. It is clear that if we are not in the generic case (see the definition for "genericity" above), there may not be a common resonance to all traveling waves and such a wavenumber k_0 (resp. a frequency ω_0) does not exist. In the generic case, our analysis implies that k_0 (resp. ω_0) must be a multiple of $\bar{q}=\prod_i q_i$ (resp. $\bar{p}=\prod_i p_i$), which we cannot determine before investigating the set of all velocities.

Although it is possible to "scan" the wavenumbers (resp. frequencies) and check the corresponding frequencies $\omega(k)$ [resp. wavenumbers $k(\omega)$], it seems difficult to account for *all* traveling waves without knowing their number *a priori*. The success of such a technique is thus limited to very special situations. In contrast, the biorthogonal decomposition, combined with the analysis performed in this study, will provide a reliable algorithm in both generic and nongeneric cases. In the case of continuous spectra, any spatial (resp. temporal) Fourier transform $\hat{u}(k, t)$ [resp. $\hat{u}(x, \omega)$] may contain all the frequencies (resp. wavenumbers). In the next section, we will see that wavepacket resonances are present in this case and therefore, for any practical purpose, the difficulty underlined above remains.

5. DISPERSIVE WAVES WITH DISCRETE SPECTRA

In this section, we mention an extension of the results of the two previous sections to a more general situation where the waves $u_1,..., u_N$ are no longer uniformly traveling waves. In fact, we used the traveling wave properties only in Eq. (4.6) defining resonances.

Let us now suppose that each wave $u_d(x, t)$ has a (Fourier) dispersion relation given by a more general function that the linear one of Eq. (3.6), namely

$$\omega_i = c_d(k_j) \tag{5.1}$$

Since we are interested in studying the superposition of such waves, we can suppose, without loss of generality, that each function c_d is an isomorphism, namely it has only one univalued branch in the (ω, k) plane. In the case of several branches, one can consider that the wave is the sum of different waves, one for each branch. The case where a branch is multivalued can be locally reduced, except in a few pathological situations, to a univalued branch and then again each local contribution can be treated as an individual wave.

Now, it is easy to realize that (5.1) also characterizes the support of the spectral measure of U_d. Therefore, the only modification required here consists in replacing the equations (4.6) defining a resonance by the new equations (5.1). Then, the results of Sections 3 and 4 still hold.

Nevertheless, the inverse problem of reconstructing the various waves is very delicate in practice, the reason being that the functions c_d are not known *a priori* and their reconstruction from the knowledge of the resonances can be a difficult puzzle.

6. THE CASE OF A CONTINUOUS SPECTRUM

It is clear that even in the case of finite domains we cannot avoid from a rigorous point of view the case where continuous spectra appear. Indeed, the ratio between the velocities may be irrational or the velocities may be irrationally related to the spatial and temporal sizes of the domain. Here the assumption of a discrete spectrum will systematically lead to the wrong conclusion that there is no resonance since the arithmetic condition between the frequencies and wavenumbers will never be satisfied. It will be then concluded that the two waves do not interact or, equivalently, that the corresponding operators commute, which obviously does not make sense.

If U_1 and U_2 are two traveling waves with continuous spectra, it is known that the notion of "wavepacket" may be introduced for handling this situation. We will see that the spectral analysis of U_1, U_2, and $U_1 + U_2$ naturally leads to the introduction of such a notion.

First, we recall that for a single uniformly traveling wave $u(x, t)$ of speed c with a continuous spectrum a spectral representation of

$$V = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} \tag{6.1}$$

is given by

$$\hat{V} = \begin{pmatrix} F & 0 \\ 0 & \tilde{F} \end{pmatrix} \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & \tilde{F} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & FU^*\tilde{F}^{-1} \\ \tilde{F}UF^{-1} & 0 \end{pmatrix} \tag{6.2}$$

In such a representation, $\hat{V}$ is a multiplication operator since we can consider the vectors $\hat{\varphi} \oplus \hat{\psi}$ as functions of (k, ω). The generalization of the analysis of Sections 3 and 4 can then be performed quite easily as follows. For every Borel set δ in K, let $\tilde{\delta} = c\delta$ be the corresponding Borel set in Ω. We then consider the corresponding projections P_δ and $P_{\tilde{\delta}}$ on the set of functions with support in δ and $\tilde{\delta}$.

In case where δ and therefore also $\tilde{\delta}$ are two (small) intervals, we call such a pair of projections a wavepacket, for obvious reasons. It is clear from (6.2) and the fact that $\hat{V}$ is a multiplication operator if u is a traveling wave that the corresponding spectral projections of $\hat{V}$ are of the form[36]

$$\begin{pmatrix} P_\delta & 0 \\ 0 & P_{\tilde{\delta}} \end{pmatrix} \tag{6.3}$$

and therefore they are generated by wavepackets. Now, the results of Sections 3 and 4 can be extended by replacing delta functions by wavepackets. Again, the key notion for studying the superposition of traveling waves is that of resonances.

We now demonstrate how it works for the superposition of two traveling waves since the generalization to more than two traveling waves is straightforward. Our goal here is to construct invariant subspaces for $U_1 + U_2$ in order to evaluate the space-time dynamics in these subspaces by an approximate dynamics with a discrete spectrum. As expected, the interaction of U_1 and U_2, which appears in the noncommutativity of the operators or equivalently in the "mixture" of their spectral projections, leads to a spreading of the wavepackets along the resonances. We now formulate these ideas in a precise way.

Definition 6.1. The resonance generated by the wavepacket $(\delta, \tilde{\delta})$, where δ is an interval in K and $\delta, \tilde{\delta} = c_i\delta$, $i = 1$ or $i = 2$, the corresponding interval in Ω is the two-side infinite sequence

$$\ldots \tilde{\delta}_{n-1}, \delta_{n-1}, \tilde{\delta}_n, \delta_n, \ldots \tag{6.4}$$

n going from $-\infty$ to $+\infty$, where

$$\tilde{\delta}_n = c_i \delta_n \tag{6.5}$$

$$\tilde{\delta}_{n+1} = c_j \delta_n \tag{6.6}$$

with $j = 1$ if $i = 2$ and $j = 2$ if $i = 1$. The nonoverlapping condition

$$\delta_{n-1} \cap \delta_n = \varnothing \tag{6.7}$$

$$\tilde{\delta}_{n-1} \cap \tilde{\delta}_n = \varnothing \tag{6.8}$$

for all n is

$$\frac{c_1 - c_2}{c_1 + c_2} \omega_0 > \varepsilon \tag{6.9}$$

for a resonance generated by a wavepacket centered on ω_0 and of radius ε. Of course, if at least one velocity is negative, (6.9) should be written with absolute values. In any case, we see that it is always possible to build a nonoverlapping resonance from a wavepacket centered at any ω_0, provided that the radius ε is sufficiently small, keeping the spreading also small compared with the distance between two consecutive wavepackets in the resonance.

We now express the equivalent of Theorem 3.5.

Theorem 6.2. The resonance generated by a wavepacket $\tilde{\delta} = \tilde{\delta}_0$ and $\delta = \delta_0$ defines the smallest subspaces,

$$P_{\tilde{\delta}} = \sum_n P_{\tilde{\delta}_n} \qquad \text{and} \qquad P_{\delta} = \sum_n P_{\delta_n}$$

invariant by $\hat{U}_1 + \hat{U}_2$ and containing the given wavepacket.

Proof. The proof is easy and makes use of functional calculus.[36]

Now, we consider a wavepacket such that condition (6.9) is fulfilled so that the corresponding resonance is made of wavepackets which do not overlap in K and Ω but, of course, interact because of the chain rule (6.5) and (6.6).

Because of the accumulation of wavepackets at zero frequency and wavenumber ($\omega = 0$, $k = 0$) and the spreading for large frequencies and wavenumbers, we cannot in general approximate a wavepacket resonance by a "corresponding" resonance with a discrete spectrum for which the results of Section 3 are available. In order to do this, some infrared and ultraviolet cutoffs need to be performed. Usually, the infrared cutoff is justified by a finite-size argument and the ultraviolet one by a fast decay of the Fourier coefficients of u_1 and u_2. Note, however, that the validity of such assumptions depends on the particular problem considered.[37] But if these cutoffs are justified, each traveling wave can be replaced by one, with a discrete spectrum, whose shape is defined by the amplitudes at each central point k_n of the wavepackets. The latter are computed by a normalized integral of the continuous amplitude over the width of the resonance. We then obtain two traveling waves satisfying the results of Section 3.

This approximation is related to Weil's criterion,[36] from which a continuous spectrum may be evaluated by "approximated" eigenvalues, but we refrain from going further on this topic since it is beyond the scope of the present study.

7. BIFURCATIONS

We now address the question of the possible spatiotemporal bifurcation mechanisms which lead to the superposition of traveling waves. The terminology "bifurcation" here denotes a qualitative change in the spatiotemporal nature of the solution, an example of which is given by the appearance of an additional traveling wave in a given superposition of traveling waves, such as the transition between a traveling wave and the sum of two traveling waves. We should mention that, *a priori*, a simple change in the shape of the existing waves without influencing the spatiotemporal structure of the solution [as studied in ref. 10 by using (spatial) bifurcation theory] is not, from our viewpoint, a (space-time) bifurcation. In analogy with Poincaré's terminology regarding temporal systems (see, e.g., ref. 21), a space-time bifurcation is characterized by a lack of smoothness in $u(x, t)$ as a function of a parameter and occurs at the intersection between two branches of "stable" or "equilibrium" solutions. For this purpose, we consider that the operator U depends on a real parameter λ. The smoothness of this dependence, as well as the mathematical definition of "smoothness" we need to consider in practice, are still open issues. Although an understanding of the physical mechanisms and/or a study of the mathematical restrictions due to the existence of a known evolution equation is needed in each case, we believe that (space-time) bifurcations are generic in a certain sense, independenty of the particular conditions of the function considered. We are interested here in these generic features and we will attempt to define the general questions which need to be addressed for detecting such bifurcations. In the next section, we present experimental data where such questions seem to be indeed relevant.

The most interesting situation arises when the dependence of the operator U upon the parameter λ is continuous, even differentiable, but nonholomorphic.[25] In this case, there are two ways in which a new velocity, and therefore a new traveling wave, may appear in a given superposition of traveling waves.

The first one corresponds to the bifurcation parameter values at which there is (at least) one degeneracy of eigenvalues which we cannot remove by small perturbations in the complex plane. We denote by λ_c one of these values. This situation, which is reminiscent of what is called "phase defects" in two-dimensional patterns,[15] can be well understood from the following geometrical point of view. For the parameter values λ close to λ_c but such that $\lambda < \lambda_c$, the eigenvectors are uniquely defined, since the eigenvalues are nondegenerate and it follows from the general perturbation theory of linear operators that they will smoothly evolve as λ varies. For $\lambda = \lambda_c$, the subspace corresponding to the degenerate eigenvalue is spanned by equivalent eigenvectors. Therefore, it may happen that in passing through λ_c a jump is observed in the eigenvectors due to their rotation in the corresponding subspace. We recall that this general perturbation theory is applicable to the operator

$$V = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} \tag{7.1}$$

and thus this rotation may result in a new linking between the topos and the chronos, for instance,

$$\begin{pmatrix}\varphi_1(x)\\ \psi_1(t)\end{pmatrix}, \begin{pmatrix}\varphi_2(x)\\ \psi_2(t)\end{pmatrix} \quad \text{for} \quad \lambda < \lambda_c \tag{7.2}$$

and

$$\begin{pmatrix}\varphi_1(x)\\ \psi_2(t)\end{pmatrix}, \begin{pmatrix}\varphi_2(x)\\ \psi_1(x)\end{pmatrix} \quad \text{for} \quad \lambda > \lambda_c \tag{7.3}$$

but other rotations are possible. It is well known (see ref. 25 or ref. 1 for a related problem) that such a jump is related to a cut in the complex plane of the extended parameter. (This cut is the projection in the complex plane of the intersection of the eigenvalue surfaces.) Therefore, in this case it is impossible to go from $\lambda < \lambda_c$ to $\lambda > \lambda_c$ on a path which avoids the described jump. In contrast, when a jump is not observed on both sides of an eigenvalue degeneracy such a path can be avoided in the complex plane due to the fact that the associated singularity (corresponding to the projection of the intersection of the eigenvalue surfaces) does not cut the complex plane into two parts. Hereafter we refer to this situation as the self-avoiding case.

In the case of propagating phenomena treated in this paper, a new velocity is clearly created at a (spatiotemporal) bifurcation point which, from the analysis of Sections 3 and 4, can be detected through the analysis of new resonances. From a practical point of view, one can distinguish between a bifurcation and a self-avoiding situation, both possible when a fourth-order degeneracy of eigenvalues (the second-order degeneracy mentioned above is not considered here) occurs, by studying the persistence of resonances as the parameter increases further.

As an example, a second velocity c_2 may emerge from a traveling wave of velocity c_1 via the previous (bifurcation) mechanism, leading to the birth of the resonance ω, k, ω', k' which are the frequencies and wavenumbers of the two crossing topos and chronos, where $\omega/k = \omega'/k' = c_1$ and $\omega/k' = c_2$. Note that a situation where a chrono-topo pair corresponding to the largest eigenvalue before λ_c corresponds to the smallest eigenvalue after λ_c is a self-avoiding case where no rotation in the subspaces has taken place and no new velocity is generated. An illustration of this possibility will appear in the second example (for $\lambda = 1$) and third example of the next section.

The second way in which a new velocity may appear from a given superposition of traveling waves as a parameter λ varies is by a direct growth of a new wave of very small amplitude. Note that this case may be considered as a particular case of the previous one by considering the zero eigenvalue of the initial operator V (for $\lambda < \lambda_c$). It makes sense, however, to treat this case separately because the new eigenvalues have small amplitudes, and therefore are difficult to find (in practice, no degeneracy can be observed among the computed eigenvalues). Nevertheless, the appearance of a new velocity in this case will be detectable by the trace of the resonances created by this new velocity on the large-energy topos and chronos.

We note that in this first case where a crossing of nonzero eigenvalues occurs the newly formed traveling wave is born with a nonzero amplitude, while in the second case the new traveling wave has zero energy at its birth. Although a bifurcation takes place in both situations, the difference between the two seems to be close to that existing between subcritical and supercritical bifurcations in temporal dynamical systems theory. Nevertheless, the formulation of an unstable branch in the first case is not clear from our study.

Moreover, we would like to point out that there is an interesting relation between this "microscopic" level of the bifurcations occurring between the eigenvalues (namely the energy of the various modes) permitting the rotation of the eigenmodes and a global or "macroscopic" behavior of the dynamics. Recall here that several global (biorthogonal) quantities were introduced[3] among which the most relevant one in the present context is the entropy defined as

$$H(u) = -\frac{1}{\ln N} \sum_{n=1}^{N} p_n \ln p_n \tag{7.4}$$

where

$$p_n = \frac{A_n^2}{\sum_{n=1}^{N} A_n^2} \tag{7.5}$$

Since this quantity represents the distribution of energy along the spectrum, as shown in previous work,[3] it may be used as an order parameter for detecting bifurcations which precisely occur at degenerate points in the spectrum (see ref. 3 for the detection of such points in a fluid experiment, although the connection with bifurcations was not clear there; see also our third example below). At this point, it is less obvious to us how to give a thermodynamic meaning to the entropy in the present context. In other words, whether the latter has a good statistical behavior when the number of superposed traveling waves and/or that of resonances increase to infinity remains an open question.

Finally, it may be worth mentioning that our space-time bifurcation approach is analogous to that provided by the center manifold theorem in the (classical) temporal bifurcation theory (see, e.g., ref. 21) in the sense that we restrict the stability study to the subspace, which may not be stable under small perturbations. The analogy, however, is still far from being complete, as the powerful tool of normal forms, for instance, is not yet available in our case.

8. EXAMPLES

As an example of spatiotemporal resonances, we consider the superposition of two traveling waves $u_1(x, t)$ and $u_2(x, t)$ of different shapes and velocities:

$$u(x, t) = u_1(x, t) + u_2(x, t) \tag{8.1}$$

both defined on the same spatial domain L, during a time duration of T. The individual waves u_i, $i = 1$ and $i = 2$, are defined by

$$u_i(x, t) = \sum_{k_i} \hat{g}_i(k_i) \sin\left[2\pi\left(\frac{k_i}{L}x - \frac{c_i k_i}{T}t\right)\right] \tag{8.2}$$

where the velocities (normalized with L/T) are $c_1 T/L = 1$, $c_2 T/L = 2/3$, the wavenumbers appearing in the traveling wave $u_i(x, t)$ being

$$k_i = 1, 2, 3, 4, 5, 6, 8, 9 \qquad \text{if} \quad i = 1 \tag{8.3}$$

$$k_i = 3, 6, 9, 12 \qquad \text{if} \quad i = 2 \tag{8.4}$$

and the Fourier amplitudes being defined as

$$\begin{aligned} \hat{g}_i(k_i) &= \frac{10}{k_i} \qquad \text{if} \quad i = 1 \\ &= \frac{21}{k_i} \qquad \text{if} \quad i = 2 \end{aligned} \tag{8.5}$$

Figures 1 and 2 display three-dimensional representations of the traveling waves $u_1(x, t)$ and $u_2(x, t)$, respectively, while Fig. 3 shows the same representation of the wave superposition $u(x, t)$. Now ignoring Eqs. (8.1)–(8.5), it is difficult to realize that $u(x, t)$ is indeed the superposition of two traveling waves, and, *a fortiori*, to determine their propagation speeds, shapes, and resonances. The biorthogonal decomposition, combined with the analysis of Sections 3 and 4, allows us to address this problem in a straightforward manner, as we now show.

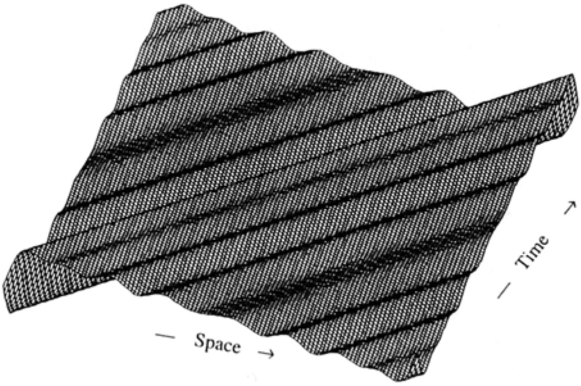

Fig. 1. Three-dimensional representation in space and time of the traveling wave $u_1(x, t)$ defined by Eqs. (8.2), (8.3), and (8.5) (see text).

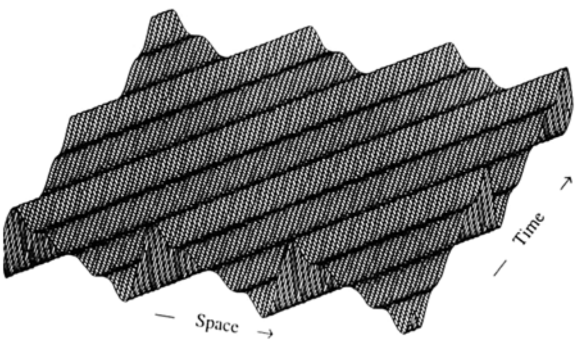

Fig. 2. Three-dimensional representation in space and time of the traveling wave $u_2(x, t)$ defined by Eqs. (8.2), (8.4), and (8.5) (see text).

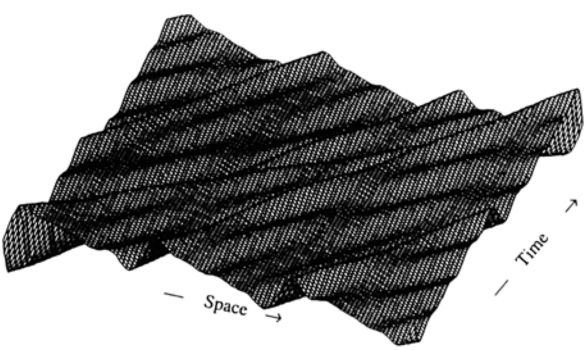

Fig. 3. Three-dimensional representation in space and time of the superposition of two traveling waves $u(x, t)$ given by Eqs. (8.1)–(8.5) (see text, Section 8).

The biorthogonal decomposition is numerically applied to $u(x, t)$ over a spatial length of 0.5 and a temporal interval of 1.02, uniformly sampled with 500 spatial points and 680 time points. The first 30 eigenvalues of the biorthogonal spectrum, displayed in Fig. 4, show that 16 eigenvalues only are nonzero and that these are (second-order) degenerate. This, however, does not lead to a conclusion regarding the full nature of the function. For this, we compute the Fourier transforms of the topos and chronos shown in Fig. 5, where we have regrouped pairs of topos and chronos with peaks at common wavenumbers and/or frequencies (only one out of the two pairs corresponding to a degenerate eigenvalue is considered, since they have the same Fourier spectra). Such groups, which we can identify as being $R_1 = \{(\psi_5, \varphi_5), (\psi_7, \varphi_7), (\psi_{15}, \varphi_{15})\}$, $R_2 = \{\psi_9, \varphi_9)\}$, $R_3 = \{(\psi_1, \varphi_1)\}$, $R_4 = \{(\psi_{11}, \varphi_{11})\}$, and $R_5 = \{(\psi_3, \varphi_3), (\psi_{13}, \varphi_{13})\}$, define resonances. R_3 and R_4 are two degenerate resonances, involving only one frequency and one wavenumber, and R_1, R_2, and R_5 are resonances of order 3, 1, and 2, respectively. The computation of the ratio between the (ordered) wavenumbers and frequencies within each resonance shows that there are only two velocities, 1 and 2/3 (normalized with L/T). Finally, the reconstruction of the traveling waves $u_1(x, t)$ and $u_2x, t)$ easily follows, as explained in Section 3.

We now illustrate how the spatiotemporal resonances described in this paper are a fundamental point for the understanding of the interaction

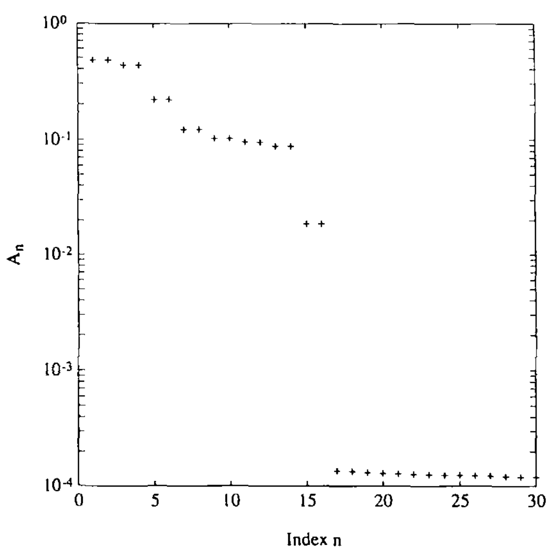

Fig. 4. Logarithm of the biorthogonal spectrum: Log(A_n) versus the mode index n.

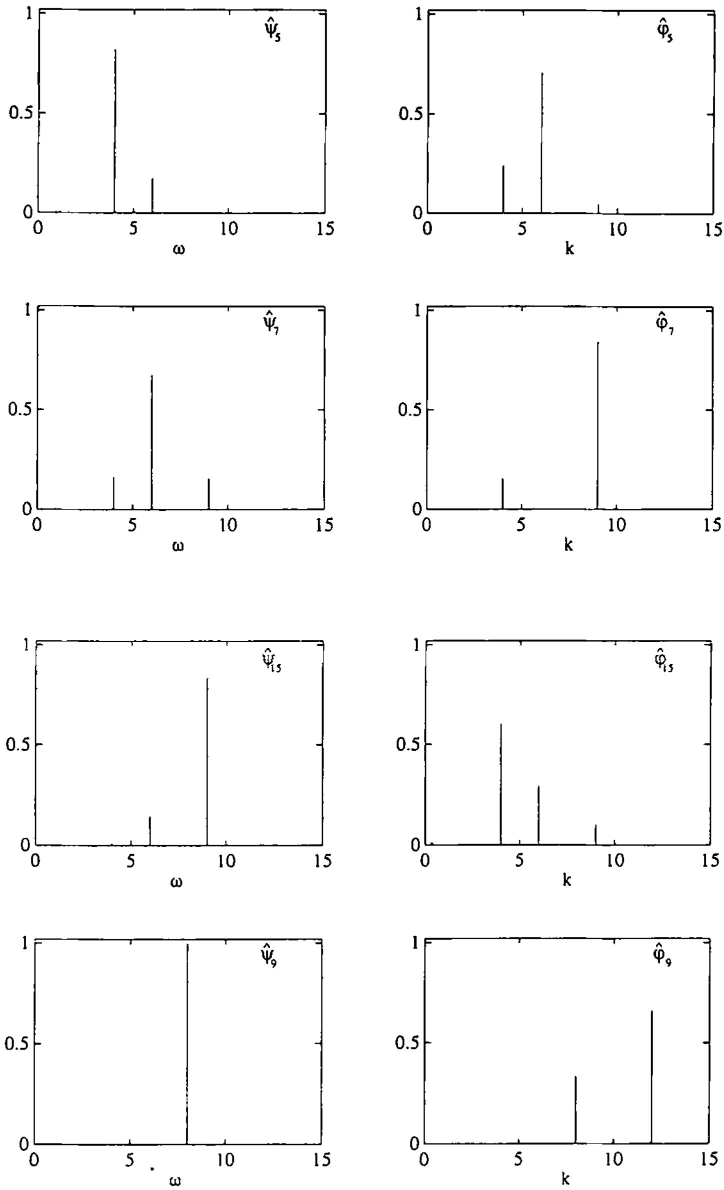

Fig. 5. Fourier spectra of the chronos (on the left-hand side) and topos (on the right-hand side). Fourier spectra of the topos and chronos belonging to the same resonance are regrouped (see Section 8). Only one chrono/topo pair corresponding to the same eigenvalue is represented, since the other pair has the same Fourier spectrum.

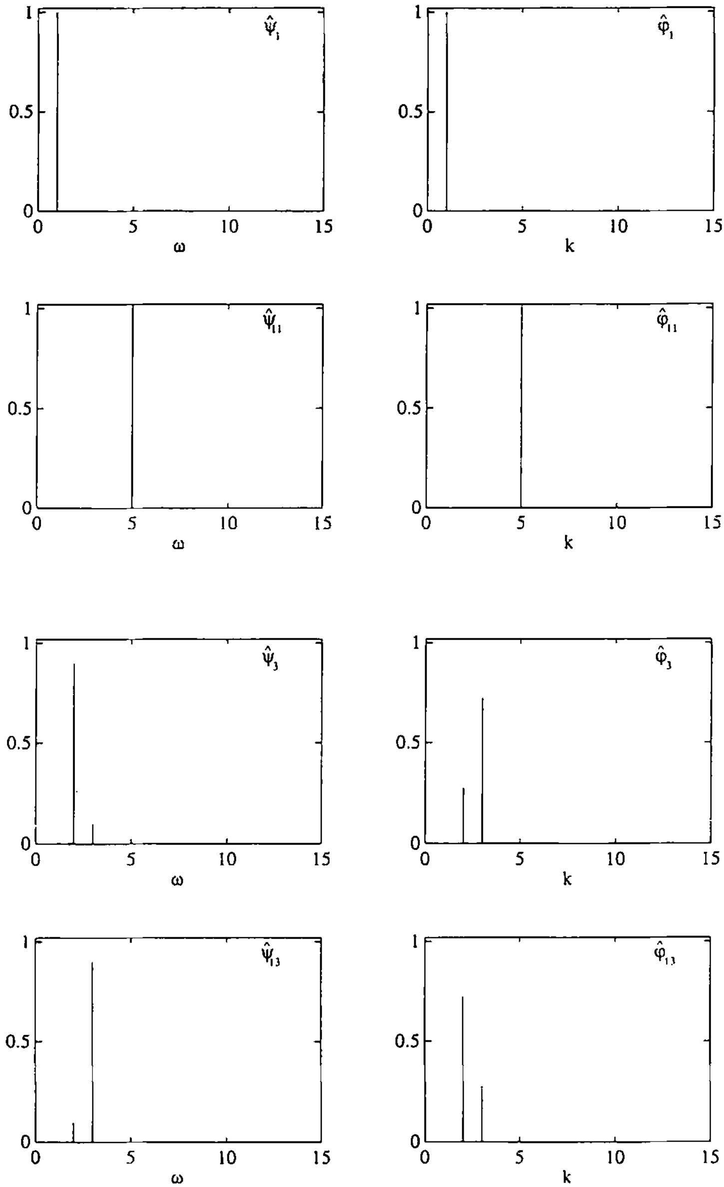

Fig. 5. (*Continued*)

between various traveling waves, which even goes beyond the specific cases treated in this paper. The fact that they córrespond to invariant subspaces of the biorthogonal operator make their extraction particularly straightforward via biorthogonal decompositions. The following questions may then arise. Is this a specific feature of the latter which fits particularly well the specific case treated in this paper, namely the superposition of various traveling waves, or is this a more general property? In other (more complex) wave propagation phenomena, can one still define the notion of resonances? In the case of a positive answer, can biorthogonal decompositions still extract them? While a general answer to these questions is difficult because of the lack of a precise form for the space-time function $u(x, t)$, we can illustrate the robustness of the method described in this paper even in cases where the hypotheses of Section 3 are no longer true. For this purpose, we consider an alternative version of Eq. (8.1), namely

$$u(x, t) = u_1(x, t) + \tilde{u}_2(x, t) \tag{8.6}$$

where $u_1(x, t)$ is the traveling wave defined by Eqs. (8.2), (8.3), and (8.5) and

$$\tilde{u}_2(x, t) = g_2(x - c_2 t - \varepsilon t^2) \tag{8.7}$$

where ε is a parameter. The function g_2 and the coefficient c_2 are the same as in the previous example [see Eq. (8.5) for g_2]. If $\varepsilon = 0$, then $\tilde{u}_2(x, t) = u_2(x, t)$ and we recover the previous case. Instead, for nonzero parameter values, $\tilde{u}_2(x, t)$ is no longer a traveling wave and its Fourier transform has nonzero coefficients for all frequencies and wavenumbers (leading to a fully broad Fourier spectrum), as a straightforward analytical computation shows. Even in this more complex situation, we now show that resonances can be directly extracted from the spectral decomposition of the operator U. In this case, we define resonances in a similar manner as in Section 3, namely as sets of frequencies and wavenumbers corresponding to closed sets of eigenequations of the spectral decomposition of the operator $V = V_1 + \tilde{V}_2$ in terms of the spectral decompositions of V_1 and $\tilde{V}_2$. Keeping this definition in mind, we can then understand the resonance structure of the dynamics, at least for $\varepsilon \leqslant 2$. We now describe our analysis for the particular case $\varepsilon = 1$ which we have numerically investigated in the same spatiotemporal domain and discretization as in the case $\varepsilon = 0$. The biorthogonal spectrum is still second-order quasidegenerate and we restrict the following description of resonances to only one mode in each pair. We first identify three resonances $\tilde{R}_a = \{(\tilde{\psi}_1, \tilde{\varphi}_1), (\tilde{\psi}_3, \tilde{\varphi}_3)\}$, $\tilde{R}_b = \{(\tilde{\psi}_5, \tilde{\varphi}_5)\}$, and $\tilde{R}_c = \{(\tilde{\psi}_{11}, \tilde{\varphi}_{11})\}$. Due to the existence of two new nonzero eigenvalues $\tilde{A}_{17}$ and $\tilde{A}_{18}$, all the remaining modes merge into one large resonance $\tilde{R}_d$ (with becomes larger as ε increases). Comparing this case to the unperturbed situation $\varepsilon = 0$, one can observe that the effect of the perturbation consists in breaking the resonances R_5 and R_1. A portion of R_5 forms the new (degenerated) $\tilde{R}_b$ and the remaining part joins R_3 to build $\tilde{R}_a$. Meanwhile, R_1 gives the new (degenerated) $\tilde{R}_c$ on the one hand, and joins R_4 and R_2 to lead to the formation of $\tilde{R}_d$ on the other hand. We would like to conclude that this example shows how the biorthogonal analysis, by maintaining the one-to-one correspondence between spatial and temporal orthogonal modes, permits an easy extraction of the dynamical information for the superposition of traveling waves or more complex spatiotemporal functions. While it is certainly true that a space-time Fourier analysis does not lose any information, in general, it is not adapted to the space-time dynamics. One may wonder, for instance, whether resonances (of frequencies and wavenumbers) in the previous example still play a role. If the perturbation from the superposition of traveling waves is relatively small, the biorthogonal analysis gives a positive answer, although a more important perturbation has the effect of linking most frequencies and wavenumbers into one large resonance, indicating that (generalized) resonances should not be defined, in this more general context, with respect to Fourier modes. Finding the invariant subspaces of the biorthogonal operator will still remain a powerful tool. Moreover, our analysis of real experimental data (such as that of our third example, see below) has shown that Fourier modes of small energy which play a crucial role in resonances are hard to distinguish from the noise level of the signal in the (space-time) Fourier analysis of the latter. In contrast, their appearance in topos and chronos, which, we recall, are renormalized, is often much clearer since Fourier peaks there have a vectorial orientation significance rather than a pure energetic meaning.

Our second example illustrates the fact that a degeneracy of eigenvalues does not necessarily imply the occurrence of a bifurcation. For this, we consider the function

$$u_\lambda(x, t) = \cos[2\pi(k_1 x - k_1 c_1 t)] + \lambda \cos[2\pi(k_2 x - k_2 c_1 t)] \qquad (8.8)$$

where λ is the bifurcation parameter. When λ is zero, $\dim \chi(X) = \dim \chi(T) = 2$ (the spectrum is second-order degenerate), and $\dim \chi(T) = \dim \chi(T) = 4$ as soon as λ becomes nonzero (for $\lambda \neq 1$, the spectrum is second-order degenerate). At the parameter value $\lambda = 1$, the spectrum becomes fourth-order degenerate, but the isomorphism between topos and chronos stays such that the velocity is still c_1: there is no spatial/temporal mode crossing and therefore no bifurcation takes place.

Our third example is extracted from the biorthogonal analysis of experimental data in a thin liquid film flowing down an incline described at length in a forthcoming paper.[13] Mode crossing situations such as those described in Section 8 are ubiquitous there and we present here one example only. Accurate measurements of the film thickness $h(x, t)$ as a function of space (downstream distance) and time have been obtained via a fluorescence imaging method by Liu *et al.*[29] Details regarding the experimental apparatus and the measurements method can be found in refs. 28 and 29. These authors have shown experimentally that the initial instability is convective and that the traveling waves thus formed are noise-sustained. The influence of the noise is investigated by varying the level of periodic external forcing at the inlet. Two parameters are thus varied in the experiment, the Reynolds number and the frequency of the small-amplitude external perturbation. The data that we now present have been taken at the same (small) external frequency of 4 Hz and at Reynolds numbers $Re = 39.1$. For such a small external frequency, subharmonic instabilities of the (primary) waves have been detected by observing the formation of subharmonic peaks in the power spectra of the local (spatial) wave slope measured as a function of time two probes.[30] These peaks are said to be consistent with the coalescence of the waves in pairs as they travel, a phenomenon which roughly doubles the (spatial) period. It is, however, observed that "the period doubling is irregular both in space and time," leading to "a spatial-temporal chaos downstream."

The data set is sampled on a spatial mesh of 512 points and a temporal mesh including 600 points, the time step and the spatial step being 0.067 sec and 0.065 cm, respectively. The center of the spatial window is located at a downstream distance of 82 cm. The experimental signal as a function of space and time is represented in Fig. 6. The biorthogonal spectrum (Fig. 7) shows a second-order quasidegeneracy of the eigenvalues above a certain level of energy (the first eigenvalue is not represented here, as it is nondegenerate, corresponding to a first term where the topo coincides with the spatial average of the signal and the chrono is constant.[3,4] We note that two pairs of eigenvalues (A_6, A_7), (A_8, A_9) are relatively close to each other. The Fourier transforms of the corresponding chronos and topos are displayed in Tables I and II showing that chronos and topos $\psi_6, \varphi_6, \psi_8, \varphi_8$ are not pure Fourier modes, but that additional (small-amplitude) peaks in the power spectra are present. The two following remarks resulting from the same analysis of data recorded at a slightly smaller Reynolds number ($Re = 35.8$) support the assumption of a rotation of eigenvectors (chronos and topos) in their respective subspaces at a Reynolds number for which the fourth-order degeneracy $A_6 = A_7 = A_8 = A_9$ occurs: (1) At $Re = 35.8$, chronos ψ_6, ψ_8, and topos φ_6, φ_8 are Fourier

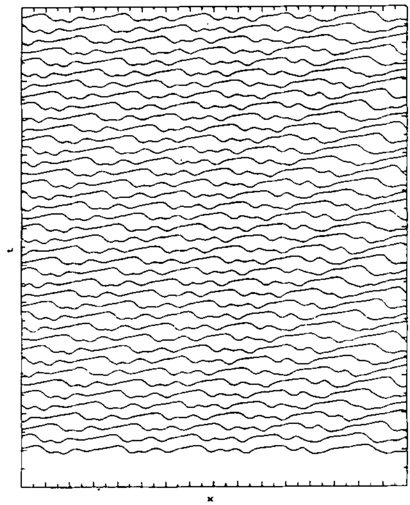

Fig. 6. Space-time representation of the experimental signal of Liu *et al.*[29] in a flowing film for a forcing frequency of 4 Hz at Reynolds number $Re = 39.1$ (only 40 time steps are represented).

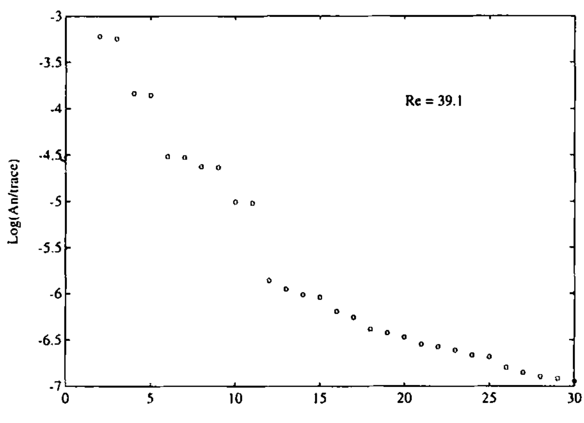

Fig. 7. Logarithm of the (renormalized) biorthogonal spectrum: $\mathrm{Log}(A_n/\sum_n A_n)$ versus n for the experimental signal of Fig. 6.

Table I. Frequencies (ω_i) in Hz and Amplitudes (a_i) of the Fourier Transforms of the Chronos for $n = 2, ..., 9$[a]

	ω_1	a_1	ω_2	a_2
$\hat{\psi}_2, \hat{\psi}_3$	4	0.49		
$\hat{\psi}_4, \hat{\psi}_5$	8	0.49		
$\hat{\psi}_6, \hat{\psi}_7$	12	1.11×10^{-2}	16	0.48
$\hat{\psi}_8, \hat{\psi}_9$	12	0.47	16	1.54×10^{-2}

[a] Peaks occur at the forcing frequency, 4 Hz, and its subharmonics.

modes with peaks located at 12 and 0.45 for ψ_6 and φ_6, respectively, and 16 and 0.57 for ψ_8 and φ_8, respectively; (2) at $Re = 35.8$, extra Fourier peaks in these topos and chronos could not be detected above the (estimated) noise level. This rotation, however, is consistent with the two phenomena described in Section 6, namely a bifurcation (from one to two traveling waves) and a self-avoiding situation. The examination of the resonances at higher Reynolds numbers will be reported in ref. 13 to address this point, in particular regarding the error made by a relatively coarse resolution in Fourier space. Although the nature of the previous rotation of topos and chronos in eigenspaces can be fully understood from Tables I and II, which follows the development of Section 7, the rotation itself was first detected by the global, "macroscopic" entropy function with first increases due to the eigenvalue degeneracy and then decreases (as in the fluid experiment example analyzed in ref. 3). This abrupt variation of the entropy as a function of the control parameter is described in more detail in ref. 13.

Table II. Wavenumbers (k_i) in cm^{-1} and Amplitudes (b_i) of the Fourier Transforms of the Topos for $n = 2, ..., 9$[a]

	k_1	b_1	k_2	b_2
$\hat{\varphi}_2, \hat{\varphi}_3$	0.15	0.43		
$\hat{\varphi}_4, \hat{\varphi}_5$	0.30	0.24		
$\hat{\varphi}_6, \hat{\varphi}_7$	0.45	1.07×10^{-2}	0.57	0.21
$\hat{\varphi}_8, \hat{\varphi}_9$	0.45	0.26	0.57	3.1×10^{-2}

[a] Peaks occur at $k_0 \approx 0.15\ cm^{-1}$ and its subharmonics, within the error due to the discretization in Fourier space.

9. CONCLUDING REMARKS

To conclude, we have shown that biorthogonal decompositions can characterize space-time dynamical behavior in wave propagation phenomena in an efficient way. Although such an expansion corresponds to a spatiotemporal Fourier decomposition in the case of a uniformly traveling wave, this is not the case as soon as (at least) one additional (pure) traveling wave (of different velocity) gets superposed to the first one. Such a superposition occurs in various situations, such as in the N-soliton solutions of the Korteweg–de Vries equation and in the stable standing wave solution (which can be considered as the sum of two traveling waves propagating at velocities c and $-c$) of two coupled Ginzburg–Landau equations (15) also arising in Taylor–Couette flow.[18] One could argue that biorthogonal modes (topos and chronos) are then still very close to Fourier modes and finding them should not be worth the trouble.... Our investigations have revealed that the spatiotemporal structure of such a solution lies precisely in the difference between biorthogonal and Fourier modes, due to the presence of (spatiotemporal) resonances (namely interactions between the various traveling waves). Resonances appear as invariant subspaces under the action of the biorthogonal operator, which defines closed sets of equations, so that topos and chronos can be classified according to the specific resonance to which they belong. Such a technique thus permits the identification of the various traveling waves (namely their velocity, their amplitude, and their shape) present in the solution.

In comparison with previous work using other techniques, we would like to emphasize that our procedure does not make any assumption on the shape and velocity of the various traveling waves, the number of the traveling waves and/or resonances included, the linearity or nonlinearity of the (dynamical) processes involved, and the nature of the resonances. For instance, regarding the latter, resonant interactions can occur globally or locally in spectral space and globally or locally in (space-time) physical space. The existence of spatiotemporal resonances, which may be spread over a wide range of frequencies and wavenumbers, as found in this paper, should have implications on our fundamental understanding of turbulence for which a basic issue is whether wavenumber interactions are local in Fourier space or not.[45] Moreover, many studies have concentrated on global resonant wave interactions, an assumption which is obviously not valid in case of strong inhomogeneities and locally interacting events. The propagation of internal gravity waves in a stratified shear flow where both the stratification and the shear vary in space and time (treated in ref. 20 for resonances between three waves) furnishes an example.

Moreover, our technique, based on biorthogonal decompositions, permits the prediction and analysis of (space-time) bifurcation events through which the qualitative spatiotemporal nature of the solution changes as a function of a parameter. These occur when two (or more) eigenvalues of the biorthogonal operator are degenerate (see the early detection of such possibility in ref. 3). This property implies that as the eigenvalues merge, the global, "macroscopic" entropy locally increases as the control parameter is varied. This point is extremely interesting, as it connects the "microscopic" spatiotemporal behavior of the solution to its "macroscopic" properties. The restriction of the study of the bifurcation to the degenerate subspace, together with the existence of the one-to-correspondence between topos and chronos (intrinsic to biorthogonal decompositions), permit the prediction of the nature of the new solution. For instance, a traveling wave can undergo a transition to two traveling waves whose shapes, velocities, and amplitudes can be computed from those present slighly before the bifurcation. A spectrum degeneracy, however, does not necessarily lead to a bifurcation.

Finally, we should mention that the technique developed in this paper for propagating wave phenomena is obviously valid for other types of spatiotemporal symmetries than translations (related to traveling waves). In this case, Fourier transforms will not play any particular role, but our notion of "microscopic" (space-time) bifurcations, together with their influence on "macroscopic" (statistical) properties such as the entropy, will still apply.

ACKNOWLEDGMENTS

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